SECTION 8.5: PARTIAL FRACTION DECOMPOSITION

IDEA: Reverse the process by which we find common denominators to add fractions to resolve a rational function into 'partial' fractions to help us integrate rational functions. For instance, knowing

$$\frac{1}{x^2-25} = \frac{1}{10}\left(\frac{1}{x-5}\right) - \frac{1}{10}\left(\frac{1}{x-5}\right)$$

enables us to more easily integrate:

$$\int \frac{1}{x^2-25}\,dx = \ldots = \frac{1}{10}\int \frac{1}{x-5}\,dx - \frac{1}{10}\int \frac{1}{x+5}\,dx = \frac{1}{10}\ln|x-5| - \frac{1}{10}\ln|x+5| + C$$

STRATEGY: We need some big tools from algebra to help us - among them the theorem that every polynomial function (with real number coefficients) can be factored as a product of linear factors along with 'irreducible' quadratics.[1] An example of the **form** of a partial fraction decomposition is given below Refer to the textbook (or a similarly reliable source) for the exact statement of the partial fractions theorem.

$$\frac{3x^2-4x+1}{x^2(x+5)(2x+1)^3(x^2+x+1)(4x^2+3)^2} = \ldots$$

$$= \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x+5} + \frac{D}{2x+1} + \frac{E}{(2x+1)^2} + \frac{F}{(2x+1)^3} + \frac{Gx+H}{x^2+x+1} + \frac{Jx+K}{4x^2+3} + \frac{Lx+M}{(4x^2+3)^2}$$

In general, each linear factor (and repeated linear factor) in the denominator can be split off with its own term with a constant numerator. Each irreducible quadratic factor (and repeated irreducible quadratic factor) in the denominator can be split off with its own term with a generic linear function as its numerator. Note in the above example, $x^2 = (x-0)^2$ is a repeated linear factor: $x^2 = (x-0)(x-0)$.

STEPS TO INTEGRATE A RATIONAL FUNCTION:

1. If the degree of the numerator is greater or equal to the degree of the denominator, divide.
2. Factor the denominator into a product of linear and irreducible and quadratic factors.
3. Write the **form** of the partial fraction decomposition.
4. Determine the unknown coefficients and proceed to integrate.

NOTE: In theory, every rational function can be integrated!

EXAMPLE 1: Find the following integrals.

1. $\displaystyle\int \frac{4x^2-9x+3}{x^3-2x^2+x}\,dx$ Ans: $3\ln|x| + \ln|x-1| + \dfrac{2}{x-1} + C$

2. $\displaystyle\int \frac{4t^2+12}{t^4+4t^2}\,dt$ Ans: $-\dfrac{3}{t} + \dfrac{1}{2}\tan^{-1}\left(\dfrac{t}{2}\right) + C$

3. $\displaystyle\int \frac{2x^2+8x+13}{x^2+4x+7}\,dx$ Ans: $2x - \dfrac{1}{\sqrt{3}}\tan^{-1}\left(\dfrac{x+2}{\sqrt{3}}\right) + C$

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[1] That is, quadratic functions with no real zeros.

EXAMPLE 2: (VIDEO) Use the Heaviside Method to help you integrate the following.

1. $\displaystyle\int \frac{5-5x}{2x^2+3x-2}\,dx$ Ans: $\displaystyle\frac{1}{2}\ln|2x-1|-3\ln|x+2|+C$

2. $\displaystyle\int \frac{x^2+3}{x(x-1)\left(x^2+1\right)}\,dx$ Ans: $\displaystyle -3\ln|x|+2\ln|x-1|+\frac{1}{2}\ln\left|x^2+1\right|-\tan^{-1}(x)+C$

3. $\displaystyle\int \frac{2x^2-3x+2}{x^3-x^2}\,dx$ Ans: $\displaystyle\ln|x|+\frac{2}{x}+\ln|x-1|+C$

EXAMPLE 3: (VIDEO) Find the following integrals.

1. $\displaystyle\int \frac{1}{x\sqrt{4-x}}\,dx$ Ans: $\displaystyle\frac{1}{2}\ln\left|\sqrt{4-x}-2\right|-\frac{1}{2}\ln\left|\sqrt{4-x}+2\right|+C$

2. $\displaystyle\int \frac{1}{e^{2t}+1}\,dt$ Ans: $\displaystyle t-\frac{1}{2}\ln\left(e^{2t}+1\right)+C$

HOMEWORK: Section 8.5: 5 - 67 odd, 77 - 83 odd